GCSE Maths – Geometry and Measures

Geometric Arguments and Proof (Higher)

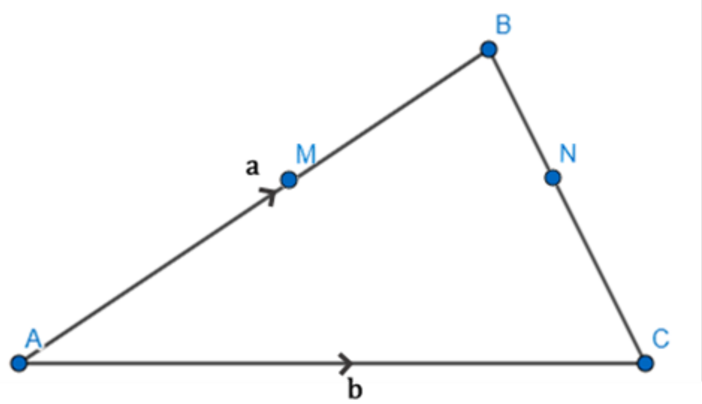
Worksheet

This worksheet will show you how to work out different types of geometric arguments and proof questions. Each section contains a worked example, a question with hints and then questions for you to work through on your own.

Section A

Worked Example

In triangle ABC, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$. M and N are the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively. Prove that $\overrightarrow{MN}$ is parallel to $\overrightarrow{AB}$.



Step 1: Find $\overrightarrow{AM}$ and $\overrightarrow{CN}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$$\overrightarrow{CB} = -\overrightarrow{AC} + \overrightarrow{AB} = -\boldsymbol{b} + \boldsymbol{a}$$

M is halfway along $\overrightarrow{AB}$ so:

$$\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB}) = \frac{1}{2}\boldsymbol{a}$$

N is halfway along $\overrightarrow{CB}$ so:

$$\overrightarrow{CN} = \frac{1}{2}(\overrightarrow{CB}) = \frac{1}{2}(-\boldsymbol{b} + \boldsymbol{a})$$

Step 2: Use these vectors to find the vector $\overrightarrow{MN}$.

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AC} + \overrightarrow{CN} = -\overrightarrow{AM} + \overrightarrow{AC} + \overrightarrow{CN} = -\left(\frac{1}{2}\boldsymbol{a}\right) + \boldsymbol{b} + \frac{1}{2}(-\boldsymbol{b} + \boldsymbol{a}) = \frac{1}{2}\boldsymbol{b}$$

Step 3: Show that $\overrightarrow{MN}$ is a multiple of $\overrightarrow{AC}$ and conclude what this means.

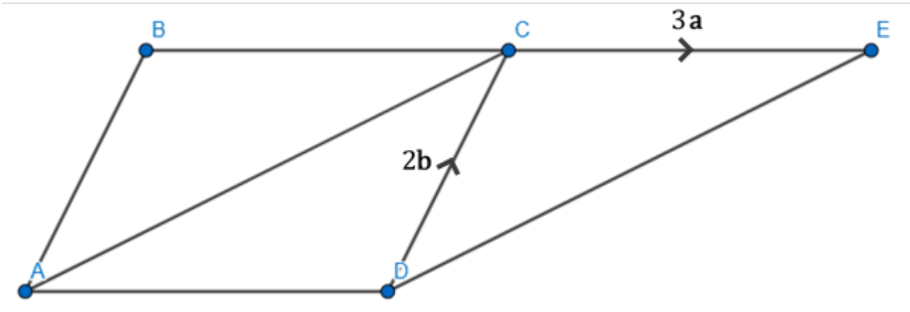
$$\left(\frac{1}{2}\boldsymbol{b}\right) \div \boldsymbol{b} = \frac{1}{2} \text{ (a constant)}$$

We have shown that $\overrightarrow{MN}$ is a constant multiple of $\overrightarrow{AC}$ and therefore $\overrightarrow{MN}$ and $\overrightarrow{AC}$ are parallel.

Guided Example

ABCD is a parallelogram, and $\overrightarrow{\mathbf{BC}} = \overrightarrow{\mathbf{CE}}$.
$\overrightarrow{\mathbf{CE}} = 3\mathbf{a}$ and $\overrightarrow{\mathbf{DC}} = 2\mathbf{b}$.

Prove that $\overrightarrow{\mathbf{AC}}$ and $\overrightarrow{\mathbf{DE}}$ are parallel lines.



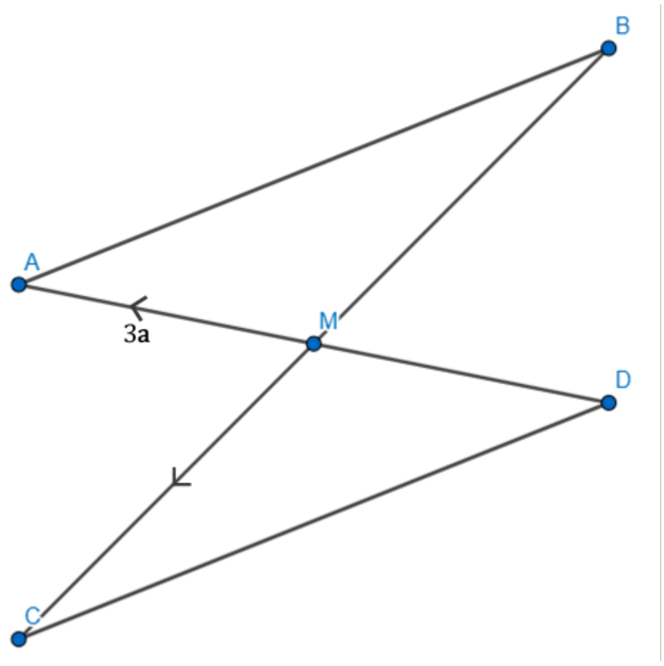
Step 1: Find the vectors $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of **a** and **b**.

Step 2: Use these to find the vector $\overrightarrow{DE}$.

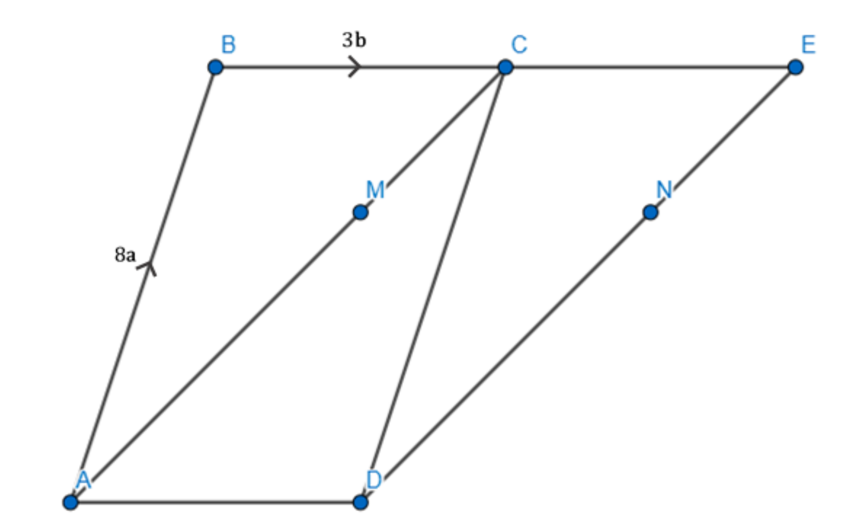
Step 3: Show that $\overrightarrow{AC}$ is a multiple of $\overrightarrow{DE}$ and conclude what this means.

Now it's your turn!
If you get stuck, look back at the worked and guided examples.

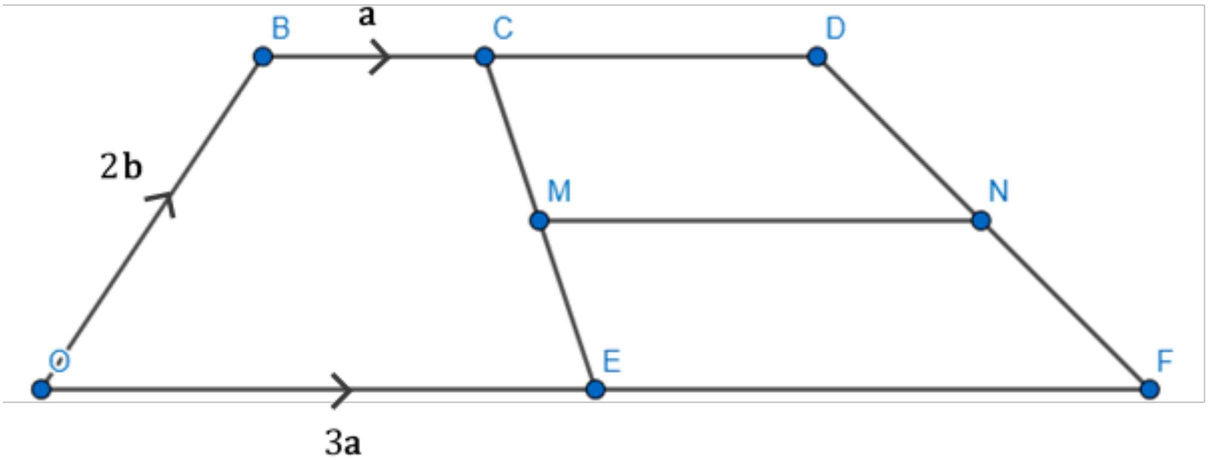
1. Two triangles AMB and DMC are congruent. M is the midpoint of $\overrightarrow{\mathrm{BC}}$ and $\overrightarrow{\mathrm{DA}}$. $\overrightarrow{\mathrm{MA}} = 3\mathbf{a}$ and $\overrightarrow{\mathrm{MC}} = 3\mathbf{b}$. Prove that $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{CD}}$ are parallel.



2. ABCD is a parallelogram.
$\overrightarrow{AB} = 8\mathbf{a}$ and $\overrightarrow{BC} = 3\mathbf{b}$.
$\overrightarrow{CE}$ is an extension of $\overrightarrow{BC}$ such that $\overrightarrow{BC} = \overrightarrow{CE}$.
M splits $\overrightarrow{AC}$ such that $\overrightarrow{AM} : \overrightarrow{MC} = 3 : 1$ and N splits $\overrightarrow{DE}$ such that $\overrightarrow{DN} : \overrightarrow{NE} = 3 : 1$.
Prove that $\overrightarrow{AD}$ and $\overrightarrow{MN}$ are parallel.



3. OBDF is a trapezium.
$\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$ and $\overrightarrow{OE} = 3\mathbf{a}$.
C is the midpoint of $\overrightarrow{BD}$ and E is the midpoint of $\overrightarrow{OF}$.
M is the midpoint of $\overrightarrow{CE}$ and N is the midpoint of $\overrightarrow{DF}$.



a) Find the vector $\overrightarrow{CE}$ in terms of **a** and **b**.

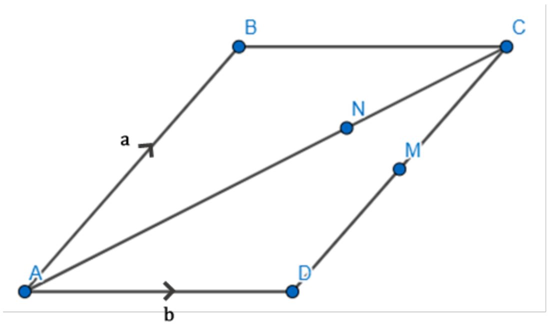
b) Find the vector $\overrightarrow{DF}$ in terms of **a** and **b**.

c) Prove that $\overrightarrow{MN}$ is parallel to $\overrightarrow{OF}$.

Section B

Worked Example

The shape ABCD is a parallelogram. $\overrightarrow{AB}$ is represented by the vector a and $\overrightarrow{AD}$ is represented by the vector b. M is the midpoint of $\overrightarrow{BC}$ and N is a point on $\overrightarrow{AC}$ such that $\overrightarrow{AN} : \overrightarrow{NC} = 2 : 1$. Prove that the points B, N and M lie on a straight line.



Step 1: Find simplified expressions for the vectors $\overrightarrow{AN}$, $\overrightarrow{AM}$, $\overrightarrow{BN}$ and $\overrightarrow{BM}$. To prove that B, N and M are on a straight line, you must show through a series of logical steps that their vectors are multiples of each other.

$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$ because the shape is a parallelogram.

Therefore, $\overrightarrow{DC} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$.

N is $\frac{2}{3}$ of the way along $\overrightarrow{AC}$, so:

$$\overrightarrow{AN} = \frac{2}{3}\overrightarrow{AC}$$

$$\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$$

$$\overrightarrow{AN} = \frac{2}{3}(\boldsymbol{a} + \boldsymbol{b}) = \frac{2}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$

M is halfway along $\overrightarrow{DC}$ so:

$$\overrightarrow{DM} = \frac{1}{2}\boldsymbol{a}$$

$$\overrightarrow{BN} = -\overrightarrow{AB} + \overrightarrow{AN} = -\boldsymbol{a} + \frac{2}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b} = -\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$

$$\overrightarrow{BM} = -\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{DM} = -\boldsymbol{a} + \boldsymbol{b} + \frac{1}{2}\boldsymbol{a} = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$$

Step 2: To prove that the points lie on a straight line, show that $\overrightarrow{BM}$ *is a multiple of* $\overrightarrow{BN}$.

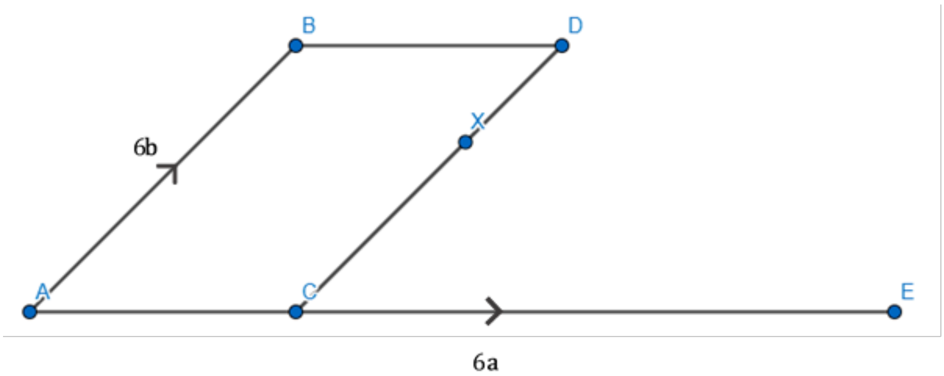
$$\frac{3}{2}\overrightarrow{BN} = \frac{3}{2}\left(-\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}\right) = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{BM}$$

Step 3: Conclude your proof.

Since $\overrightarrow{BM} = \frac{3}{2}\overrightarrow{BN}$, $\overrightarrow{BM}$ and $\overrightarrow{BN}$ are parallel. They also share a common point B so the points B, N and M lie on a straight line.

Guided Example

ABCD is a parallelogram.
$\overrightarrow{\mathrm{AB}} = 6\mathrm{b}$ **and** $\overrightarrow{\mathrm{AE}} = 6\mathrm{a}$
$\overrightarrow{\mathrm{CE}}$ **is an extension of the line** $\overrightarrow{\mathrm{AC}}$ **such that** $\overrightarrow{\mathrm{AC}} : \overrightarrow{\mathrm{CE}} = 1 : 2$**.**
The point X splits the line $\overrightarrow{\mathrm{CD}}$ **such that** $\overrightarrow{\mathrm{CX}} : \overrightarrow{\mathrm{XD}} = 2 : 1$**.**
Prove that the points B, X and E lie on a straight line.



Step 1: Deduce $\overrightarrow{\mathrm{CD}}$ and $\overrightarrow{\mathrm{BD}}$ in terms of **a** and **b**.

Step 2: Use the ratios given to find $\overrightarrow{\mathrm{DX}}$ and $\overrightarrow{\mathrm{CE}}$ in terms of **a** and **b**.

Step 3: Combine these vectors to find $\overrightarrow{\mathrm{BX}}$ and $\overrightarrow{\mathrm{XE}}$ (or $\overrightarrow{\mathrm{BE}}$) in terms of **a** and **b**.

Step 4: Write a conclusion, showing that the vectors are multiples of each other and stating the common point. Explain what this means about the points.

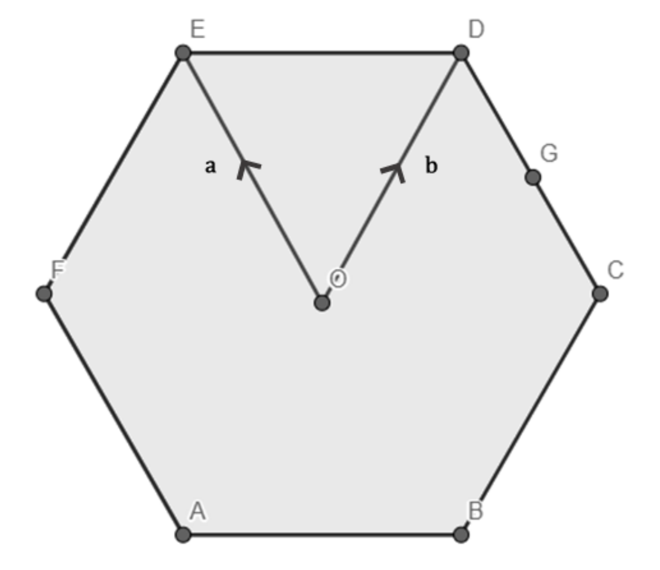
Now it's your turn!
If you get stuck, look back at the worked and guided examples.

4. ABCDEF is a regular hexagon with centre O.

   $\overrightarrow{OE} = \boldsymbol{a}$ and $\overrightarrow{OD} = \boldsymbol{b}$

   The point X lies on an extension of ED, such that $\overrightarrow{EX} : \overrightarrow{DX} = 2 : 1$.
   $\overrightarrow{EX} = -2a + 2b$

   G is the midpoint of CD.



a) Draw the vector $\overrightarrow{AO}$ and label it in terms of **a** and **b**.

b) Label all 6 sides of the hexagon in terms of **a** and **b**.

c) Find the vector $\overrightarrow{DG}$ in terms of **a** and **b**.

d) Find the vector $\overrightarrow{DX}$ in terms of **a** and **b**

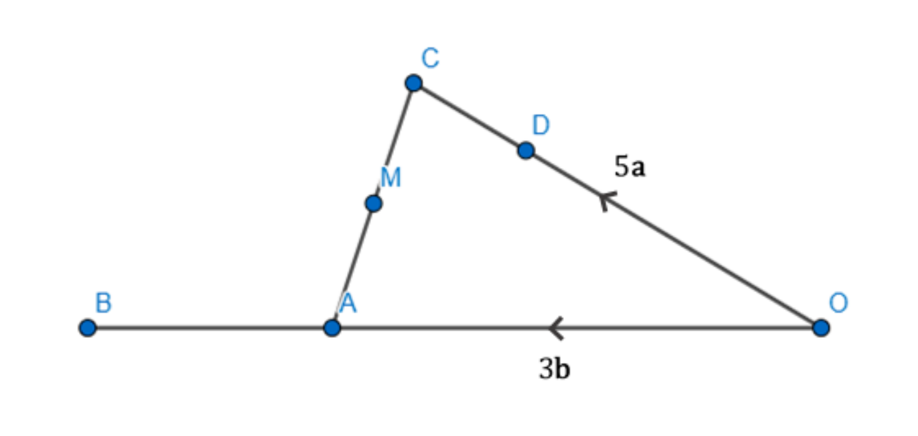
e) Hence, prove that O, G and X lie on a straight line.

5. The triangle AOC lies on a straight line OB.

$\overrightarrow{OC} = 5\boldsymbol{a}$ and $\overrightarrow{OA} = 3\boldsymbol{b}$

D is the point such that $\overrightarrow{OD} : \overrightarrow{DC} = 4 : 1$
M is the midpoint of $\overrightarrow{AC}$
B is the point such that $\overrightarrow{OA} : \overrightarrow{AB} = 3 : 1$

Show that D, M and B lie on the same straight line.



6. ABCD is a parallelogram.

$\overrightarrow{DA} = 12\mathbf{a} + 20\mathbf{b}$ and $\overrightarrow{DC} = 8\mathbf{a} + 4\mathbf{b}$.
X lies on the line $\overrightarrow{AB}$ such that $\overrightarrow{AX} : \overrightarrow{XB} = 1 : 3$
M is the midpoint of $\overrightarrow{DB}$
$\overrightarrow{CE}$ is an extension of $\overrightarrow{BC}$
Y lies on the line $\overrightarrow{CE}$ such that $\overrightarrow{CY} = -\frac{1}{2}\overrightarrow{DA}$.

Prove that X, M and Y are collinear.